Superconducting Instability of Fermions.

Above we saw that repulsive interactions at half-filling destabilize a fermi surface and lead to anti-ferromagnetic ordering.
What about attractive interactions i.e. $U < 0$. In this case mean-field theory suggests there is no anti-ferromagnetic ordering. Interestingly, now one finds a "superconducting instability.
Before we do a mean-field theory for superconductor, lets first understand very basics of a superconductor - heuristically. That will help us do mean-field theory as well.

What is a superconductor?

In a superconductor, electrons pair up to form a boson, called 'cooper pair'. (named after Sheldon Cooper from big bang theory).

The easiest way to pair them is to form a boson:

$$b(x) = c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x)$$ so that

the two electrons participating in pairing can be at the same location in the real-space.

In a superconductor, the expectation value of $b(x)$ w.r.t. to the ground state wfn does not fluctuate much, similar to the ordering of $\langle S^z \rangle$ in an antiferromagnet.

This motivates the following mean-field :

$$\langle c^\dagger_\uparrow(x)\, c^\dagger_\downarrow(x) \rangle = \Delta$$

= independent of x.

Question : How can $\langle c^\dagger_\uparrow(x)\, c^\dagger_\downarrow(x) \rangle$ be non-zero <u>if</u> the number of particles in the wfn is conserved ?

Answer : It can't but enlarging the Hilbert space to multi-particle space allows it to be non-zero.

Allowing the total number of particles to fluctuate, the wf^n can be written as a superposition of states with different no. of particles :

$$|\psi\rangle = \underbrace{|\phi_{N=1}\rangle}_{\text{1-particle wf}^n} + \underbrace{|\phi_{N=2}\rangle}_{\text{2-particle wf}^n} + \cdots |\phi_{N=\infty}\rangle$$

Within this enlarged Hilbert space, it is now possible for $\langle c^{\dagger}_{\uparrow} c^{\dagger}_{\downarrow} \rangle$ to be non-zero e.g. $\langle \phi_{N+2} | c^{\dagger}_{\uparrow}(x) c^{\dagger}_{\downarrow}(x) | \phi_N \rangle$ could in principle be non-zero.

Towards a Mean-Field Theory of a Superconductor

Again consider the same model as the one we studied for anti-ferromagnetic instability of fermi gas:

$$H = \sum_k \varepsilon_k \, c^{\dagger}_{k\sigma} c_{k\sigma} + U \sum_i (\hat{n}_i - 1)^2$$

$$n_i = \sum_i c^{\dagger}_{i\sigma} c_{i\sigma}$$

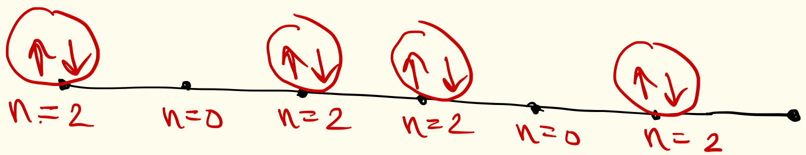
$$\varepsilon_k = -2t(\cos(k_x) + \cos(k_y)) - \mu$$

μ = chemical potential.

When U was positive, we found that anti-ferromagnet was favored. We argued this based on the limit $U \to \infty$ where $H_{eff} = \frac{4t^2}{U} \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$.

For getting a superconductor, $U < 0$ is more conducive. To see this, now consider the limit $U \to -\infty$. The term $U(n_i - 1)^2 = -|U|(n_i - 1)^2$ will favor $n_i = 0$ and $n_i = 2$.

So the ground state will look like



etc.

The bound state of two electrons at a given site is precisely the cooper pair $c^\dagger_\uparrow c^\dagger_\downarrow$ responsible for the superconductivity.

Note that, unlike the case of Antiferromagnet in the limit $U \to \infty$, the above argument <u>does not</u> rely on half-filling, and is valid for arbitrary value of the chemical potential μ.

Thus, we write H as: ($\mu' = \mu$ + constant)

$$H = \sum_{k\sigma} (\varepsilon_k - \mu') c^{+}_{k\sigma} c_{k\sigma} + U \sum_i c^{+}_{i\uparrow} c_{i\uparrow} c^{+}_{i\downarrow} c_{i\downarrow}$$

$$= \sum_{k} (\varepsilon_k - \mu) c^{+}_{k\sigma} c_{k\sigma} - U \sum_i c^{+}_{i\uparrow} c^{+}_{i\downarrow} c_{i\uparrow} c_{i\downarrow}$$

Now, we make a mean-field approximation for the interaction term:

$$-U\, c^+_{i\uparrow}\, c^+_{i\downarrow}\, c_{i\uparrow}\, c_{i\downarrow}$$

$$\longrightarrow \; -U\Big[\langle c^+_{i\uparrow}\, c^+_{i\downarrow}\rangle\, c_{i\uparrow}\, c_{i\downarrow}$$

$$+\; c^+_{i\uparrow}\, c^+_{i\downarrow}\, \langle c_{i\uparrow}\, c_{i\downarrow}\rangle$$

$$-\; \langle c^+_{i\uparrow}\, c^+_{i\downarrow}\rangle \langle c_{i\uparrow}\, c_{i\downarrow}\rangle\Big]$$

Denoting $\langle c^+_{i\uparrow}\, c^+_{i\downarrow}\rangle = \Delta =$ constant

$\Rightarrow \langle c_{i\uparrow}\, c_{i\downarrow}\rangle = -\Delta^*$

$$H_{\text{Mean-field}} = \sum_{k\sigma} (\varepsilon_k - \mu')\, c^+_{k\sigma} c_{k\sigma}$$

$\equiv H_{MF}$ henceforth

$$+\, U \sum_i \Big[\Delta\, c_{i\uparrow}\, c_{i\downarrow} + \Delta^*\, c^+_{i\downarrow}\, c^+_{i\uparrow}\Big] - N_s U |\Delta|^2$$

Note the negative sign

# of sites

$$= \sum_{k\sigma} (\varepsilon_k - \mu')\, c^+_{k\sigma}\, c_{k\sigma}$$

$$+\, U \sum_k \Big[\Delta^*\, c^+_{k\uparrow}\, c^+_{-k\downarrow} + \Delta\, c_{-k\downarrow}\, c_{k\uparrow}\Big]$$

$$-\, U|\Delta|^2 N_s$$

Note that the original Hamiltonian with $U n_{i\uparrow} n_{i\downarrow}$ term had the symmetry

$$c_\sigma \to e^{i\theta} c_\sigma$$ corresponding to the particle number conservation (recall pset-2). The mean-field H_{MF} breaks this symmetry! The only symmetry left is $c_\sigma \to -c_\sigma$ corresponding to particle no. conservation modulo two.

How to solve the mean-field H_{MF}?

Define
$$c^\dagger_{-k\downarrow} = \tilde{c}_{+k\downarrow}$$
$$c^\dagger_{k\uparrow} = \tilde{c}^\dagger_{k\uparrow}$$

Check: $\tilde{c}_{k\downarrow} \tilde{c}^\dagger_{k'\downarrow} + \tilde{c}^\dagger_{k'\downarrow} \tilde{c}_{k\downarrow}$

$$= c^\dagger_{-k\downarrow} c_{-k'\downarrow} + c_{-k'\downarrow} c^\dagger_{-k\downarrow}$$

$$= \delta_{kk'}$$ $\Rightarrow$ $\tilde{c}_{k\sigma}, \tilde{c}^\dagger_{k\sigma}$ legitimate fermion operators.

$$\Rightarrow H = \sum_k (\varepsilon_k - \mu') \tilde{c}^\dagger_{k\uparrow} \tilde{c}_{k\uparrow}$$

$$+ \sum_k (\varepsilon_k - \mu') \tilde{c}_{k\downarrow} \tilde{c}^\dagger_{k\downarrow}$$

$$- U \sum_k \left[\tilde{c}^\dagger_{k\uparrow} \tilde{c}_{k\downarrow} \Delta^* + h.c. \right]$$

$$- U |\Delta|^2 N_s$$

$$= \sum_k \left[\tilde{c}^\dagger_{k\uparrow} \quad \tilde{c}_{k\downarrow} \right] \begin{bmatrix} \varepsilon_k - \mu' & +U\Delta^* \\ +U\Delta & -(\varepsilon_k - \mu') \end{bmatrix} \begin{bmatrix} \tilde{c}_{k\uparrow} \\ \tilde{c}_{k\downarrow} \end{bmatrix}$$

$$- U |\Delta|^2 N_s$$

When $U<0$, this is identical to the mean-field Hamiltonian for the anti-ferromagnetic instability!

eigenvalues $E_k = \pm \sqrt{(\varepsilon_k - \mu')^2 + U^2 |\Delta|^2}$

Ground state energy

$$= - \sum_k \sqrt{(\varepsilon_k - \mu')^2 + U^2 |\Delta|^2}$$

$$+ |U| N_s |\Delta|^2$$

Minimizing w.r.t. $|\Delta|$, one again finds

$$|\Delta| \simeq \frac{t}{|U|} \exp\left[-1/N(\varepsilon_F)|U|\right]$$

where $N(\varepsilon_F)$ = density of states at the fermi surface,

exactly as in the case for the anti-ferromagnetic instability.

Thus, a $U > 0$ anti-ferromagnetic instability gets mapped to a $U < 0$ superconducting instability.